Information theory: assignment 4

1. Let $S \subseteq \binom{[2n]}{n}$; that is, S is a collection of subsets of $[2n]$, each of size n. Consider the following random variable T that takes values in $[2n]$. Choose X uniformly in S and choose T uniformly in X. Prove that for every $\varepsilon > 0$, there is $\delta > 0$ so that if $|S| > 2^{2n(1-\delta)}$ then the distribution of T is ε-close in statistical distance to the uniform distribution on $[2n]$.

2. Let p_t be a probability distribution of a continuous time random walk on a finite d-regular graph $G = (V, E)$. That is, let L be the Laplacian matrix of the graph defined by: for every $x, y \in V$,
$$L_{x,y} = \begin{cases} 1 & x = y \\ -\frac{1}{d} & \{x, y\} \in E \\ 0 & \text{otherwise} \end{cases}$$
Let p_0 be the initial distribution on V, and let
$$p_t = p_0 e^{-Lt}.$$
Denote by u the uniform distribution on V. Denote by $\mathcal{E}$ the *energy* operator, and denote by Ent the entropy operator; for functions $f, g : V \to \mathbb{R}_{\geq 0}$, they are defined by
$$\mathcal{E}(f, g) = \mathbb{E}_{x\sim u} f(x)(Lg)(x)$$
and
$$Ent(f) = \mathbb{E}_{x\sim u} f(x) \log \frac{f(x)}{\mu},$$
where $\mu = \mathbb{E}_{x\sim u} f(x)$. Assume that a log-sobolev inequality holds: there is a constant $\rho > 0$ so that for all $f : V \to \mathbb{R}_{\geq 0}$,
$$\mathcal{E}(f, f) \geq \rho Ent(f^2).$$
   (a) Prove that for every $f : V \to \mathbb{R}_{\geq 0}$, it holds that $\mathcal{E}(f, \ln f) \geq 2\mathcal{E}(\sqrt{f}, \sqrt{f})$.
   (b) Prove that $u = ue^{-Lt}$ for all $t \geq 0$.
   (c) Define $h_t : V \to \mathbb{R}$ by $h_t(x) = \frac{p_t(x)}{u(x)}$. Compute $\frac{d}{dt} Ent(h_t)$.
   (d) Compute $\mu = \mathbb{E}_{x\sim u} h_t(x)$.
   (e) Prove that $Ent(h_t)$ tends exponentially fast to zero as $t \to \infty$. What does it say about the behavior of the random walk?

3. Let $q = q_1 \times q_2 \times \ldots \times q_n$ be a product distribution on $\{0, 1\}^n$. We have seen the sub-additivity of entropy. Prove the sub-additivity of variance: for every non-negative function f on $\{0, 1\}^n$,
$$V_{x\sim q}(f(x)) \leq \sum_{i=1}^{n} \mathbb{E}_{x_1\sim q_1,\ldots,x_{i-1}\sim q_{i-1},x_{i+1}\sim q_{i+1},\ldots,x_n\sim q_n} V_{x_i\sim q_i}(f(x)),$$
where V stands for variance ($V(Y) = \mathbb{E}Y^2 - (\mathbb{E}Y)^2$ for a random variables Y).