Information theory: assignment 1

1. Let X, Y, Z be three Bernoulli(1/2) random variables that are pairwise independent. Under this constraint, what is the minimum value for $H(X, Y, Z)$? Give an example achieving this minimum.

   What would the minimum value for $H(X, Y, Z)$ be if the constraint above was replaced with $H(A|B) = \alpha$ for every two different random variables A, B in $\{X, Y, Z\}$ and some $0 \leq \alpha \leq 1$?

2. Let A be a subset of $\{0, 1\}^n$ with average weight at least $0.6n$. That is, if a is a (uniform) random element of A, then $\mathbb{E}|a| \geq 0.6n$, where $|a| = \sum_{i \in [n]} a_i$.

   Prove that there is a constant $c < 1$ so that $|A| \leq 2^{cn}$. What is the smallest c you can achieve?

3. (a) Prove that for every three (finitely supported) random variables X, Y, Z,

   $$2H(X, Y, Z) \leq H(X, Y) + H(X, Z) + H(Y, Z).$$

   Can you generalize this inequality? Can you prove the generalization?

   (b) For a set $A \subset \mathbb{Z}^3$, let $\pi_{23}(A)$ be the projection of A onto the 23-plane, that is,

   $$\pi_{23}(A) = \{(x_2, x_3) \in \mathbb{Z}^2 : \exists x_1 \in \mathbb{Z} \ \ (x_1, x_2, x_3) \in A\}.$$

   Define $\pi_{13}(A), \pi_{23}(A)$ similarly. What is the maximal size of a set $A \subset \mathbb{Z}^3$ so that

   $$|\pi_{23}(A)| = 12, \ |\pi_{13}(A)| = 20, \ |\pi_{23}(A)| = 15?$$

   What is the set that attains this maximum?