Introduction to combinatorics[1]

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[1]An apology: this text probably contains errors.

Contents

Chapter 1

Preface

These notes summarize an introductory course to combinatorics. Part of the goals of this course is to learn the language of mathematics, get familiar with notions and basic ideas, and improving the ability to understand and investigate abstract concepts. Remember to ask as many questions as you wish, but try to think about it a little before asking.

Chapter 2

Preliminaries

We begin with a short recollection of notions and notation.

Sets

$A = \{1, 2, 3\}$, $a \in A$, $A \subseteq B$.
$\emptyset, \mathbb{N} = \{1, 2, 3, \dots, \}, \mathbb{Z} = \{\dots, -2, -1, 0, 1, \dots\}, [n] = \{1, 2, \dots, n\}$.
$A \cap B, A \cup B, A \setminus B$, and Venn diagrams.
$A \times B = \{(a, b) : a \in A, b \in B\}$.

If you see notation that you do not understand, ask.

Induction

Induction is a framework to prove theorems, mostly on properties of $\mathbb{N}$. It is used to prove theorem of the form "for every $n \in \mathbb{N}$, we have $n \in P$" where P is some "property" of integers.

Proofs by induction have two parts: 1. "Base" in which a few preliminary cases are verified (proving that say $1, 2 \in P$). 2. "Step" in which we prove that for all $n \geq 2$ if $n \in P$ then $n + 1 \in P$.

The axiom underlying proofs by induction is: for every non-empty subset of the integers has a minimum element.

Here is an example.

Claim 1. *For every $n \in \mathbb{N}$,*
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Proof. Base: Since $1 = 1 \cdot 2/2$, the claim holds for $n = 1$. Step: Assume the claim holds for n:
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Thus,
$$\sum_{i=1}^{n+1} i = n+1 + \sum_{i=1}^{n} i = n+1 + \frac{n(n+1)}{2} = \frac{(n+1)(n+2)}{2}.$$

$\square$

More abstractly, proofs by induction are useful when we wish to prove a statement on a collection of objects, where each object can be constructed using a finite list of "composition steps" from a finite list of "basic objects". In the case of $\mathbb{N}$, every natural number satisfied $n = (n-1)+1$, so it can be constructed using the operation $+1$ from 1.

When using induction, we need to pay attention to the rules of the game. For example (Polya): All walls have the same color. "Proof": Base: this wall is white. Step: If we have n walls numbered $\{1, 2, \ldots, n\}$ then the set $\{1, 2, \ldots, n-1\}$ of walls has the same color and the set $\{2, \ldots, n\}$ has the same color. Therefore, all walls in $\{1, 2, \ldots, n\}$ have the same color. $\square$ What is mistake?

Comparing sizes

We shall use the following simple facts. Let A, B be finite sets.

- If there is a one-to-one map from A to B then $|A| \leq |B|$.

- If there is a map from A onto B then $|A| \geq |B|$.

- If there is a one-to-one and onto map from A to B then $|A| = |B|$.

Chapter 3

Counting

A large part of combinatorics is counting and enumeration; calculating or estimating the size of a given set of objects, like in how many possible outcomes to the lottery are there, or in how many ways can we pair students in a given class. In this chapter, we shall see a few basic tools for answering such questions, mostly by considering examples.

3.1   Basic methods

Cartesian products

If A, B are finite sets then

$$|A \times B| = |A| \cdot |B|.$$

More generally, by induction,

$$|A_1 \times A_2 \ldots \times A_n| = \prod_{i=1}^{n} |A_i|.$$

Bijections

What is the number of subsets of $[n]$? In other words, what is the size of $\{S : S \subseteq [n]\}$? 2^n.

Indeed, denote by P the number of subsets of $[n]$. For every $S \in P$, define $x(S) \in \{0,1\}^n$ by $(x(S))_i = 1$ iff $i \in S$. The vector $x(S)$ is called the indicator of S. The map $S \mapsto x(S)$ is a one-to-one and onto map from P to $\{0,1\}^n$. The size of $\{0,1\}^n$ is

$$|\{0,1\}^n| = |\{0,1\}|^n = 2^n.$$

Double counting

Problem 2. *In a classroom there are 32 boys, each boy knows 5 girls, and each girl knows 8 boys ("knowing" is symmetric). How many girls in class are there?*

Solution. Denote by A the set of boys, and by B the set of girls. For $a \in A$ and $b \in B$, let $f(a, b) = 1$ if a knows b, and $f(a, b) = 0$ otherwise. Thus,

$$\sum_{a \in A} \sum_{b \in B} f(a, b) = \sum_{a \in A} 5 = 5|A| = 5 \cdot 32.$$

On the other hand, the sum is also equals to

$$\sum_{b \in B} \sum_{a \in A} f(a, b) = 8|B|.$$

So,

$$|B| = 5 \cdot 4 = 20.$$

$\square$

Permutations

A permutation of $[k]$ is a one-to-one and onto map from $[k]$ to $[k]$. It is a re-ordering of k. Denote by $\Pi = \Pi_k$ the permutation on $[k]$. How many permutation of k are there? We need to decide what is $\pi(1)$. There are k options. Then we need to decide what is $\pi(2)$. There are now $k - 1$ options. And so forth. Overall,

$$|\Pi| = k(k - 1)(k - 2) \dots 1 = k!.$$

3.2   Choosing

Assume that there are n numbered balls and we want to choose k of them. There are four options:

	no repetition	repetition allowed
order matters	ordered sets of size k (2)	sequences of length k (1)
order does not matter	sets of size k (3)	multi-sets of size k (4)

1. Sequences of length k

We are interested in the size of

$$T_1 = \{(i_1, i_2, \ldots, i_k) \in [n]^k\} = [n]^k,$$

which is n^k.

2. Ordered sets of size k

We are interested in the size of

$$T_2 = \{(i_1, i_2, \ldots, i_k) \in [n]^k : \forall j \neq j' \; i_j \neq i_{j'}\}.$$

How many options for i_1? n.
After fixing i_1, how many options for i_2? $n - 1$.
After fixing i_1, i_2, how many options for i_3? $n - 2$.
Overall, the size of this set is

$$|T_2| = n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}.$$

3. Sets of size k

We are interested in the size of

$$T_3 = \{\{i_1, i_2, \ldots, i_k\} \subseteq [n]\} = \{S \subseteq [n] : |S| = k\} := \binom{[n]}{k}.$$

Now, fix $I = \{i_1 < i_2 < \ldots < i_k\} \in T_3$. For every permutation π of $[k]$, there is a sequence $(i_{\pi(1)}, i_{\pi(2)}, \ldots, i_{\pi(k)}) \in T_2$.
   The map

$$T_3 \times \Pi_k \to T_2$$
$$(I, \pi) \mapsto (i_{\pi(1)}, i_{\pi(2)}, \ldots, i_{\pi(k)})$$

is one-to-one and onto (verify). Thus,

$$|T_3 \times \Pi_k| = |T_2|$$

and

$$|T_3| = \frac{|T_2|}{|\Pi_k|} = \frac{n!}{(n-k)!k!} := \binom{n}{k}.$$

4. Multi-sets of size k

We are interested in the number of ways to choose k balls from n when order does not matter and repetitions are allowed. Such a choice is given by

$$x = (x_1, x_2, \ldots, x_n)$$

where each $x_i \geq 0$, and $\sum_{i=1}^{n} x_i = k$. The number x_i encodes the number of times i appears in the choice.

We think of x as follows. Image n boxes in a row and k bottles, where box i contains exactly x_i bottles. We can think of bottles as B's and of separator between boxes as $|$. For example, for $n = 5$ and $(1, 0, 0, 2, 1)$ with $k = 4$ we get

$$B|||BB|B.$$

In the left box there is 1 bottle, in the one to its right there are 0 bottles and so forth. So every such x corresponds to a sequence of length $n - 1 + k$ symbols, k of which are B and $n - 1$ are $|$. What is the number of such sequences? We just need to choose where to put the B's, so

$$|T_4| = \binom{n - 1 + k}{k}.$$

3.3 Binomial coefficients

We now investigate one of the most common number in mathematics: The number of sets of size k in $[n]$. We saw

$$\left| \binom{[n]}{k} \right| = \binom{n}{k}.$$

For notational convenience, if $k < 0$ or $k > 0$ we set $\binom{n}{k} = 0$; and $\binom{0}{0} = 1$. These are called the binomial coefficients, and are read "n choose k".

Inductive formula

Claim 3. *For every $n \geq 2$ and $0 \leq k \leq n$, we have*

$$\binom{n}{k} = \binom{n - 1}{k} + \binom{n - 1}{k - 1}.$$

Proof. Can be proved by direct calculation.

Here is a more explanatory proof. If $S \subseteq [n]$ is of size $|S| = k$, then there are two options: $n \in S$ and $n \notin S$. If $n \in S$ then $S \cap [n - 1]$ is a subset of $[n - 1]$ of size $k - 1$,

and if $n \notin S$ then $S \cap [n-1]$ is a subset of $[n-1]$ of size k. That is, there is a bijection between $\binom{[n]}{k}$ and the disjoint union of $\binom{[n-1]}{k}$ and $\binom{[n-1]}{k-1}$. □

Pascal's triangle

The numbers are nicely positioned in Pascal's triangle

$$010$$
$$0110$$
$$01210$$
$$013310$$
$$\ldots$$

Binomial formula

The name "binomial coefficients" comes from Newton's binomial formula:

Claim 4. *For every real x, y and $n \geq 0$,*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof. This can be proved by induction using $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$, and is left as an exercise.

In words, the coefficient of $x^k y^{n-k}$ corresponds to choosing from which of the n products we choose k copies of x. □

For example,
$$(x+y)^2 = x^2 + 2xy + y^2$$
and
$$(x+y)^3 = x^3 + 3x^2 y + 3xy^2 + y^3.$$

Symmetry

For a fixed n, the sequence $\binom{n}{0}, \binom{n}{1}, \ldots$ is symmetric in that

$$\binom{n}{k} = \binom{n}{n-k}.$$

Indeed, every choice of subset S of $[n]$ of size k yields a choice of a subset $[n] \setminus S$ of size $n-k$ (the complement) and vice versa. In other words, there is a one-to-one and onto map from $\binom{[n]}{k}$ to $\binom{[n]}{n-k}$.

Unimodality

For a fixed n, the sequence $\binom{n}{0}, \binom{n}{1}, \ldots$ is unimodal, that is, it increases and then decreases.

Claim 5. $\binom{n}{k+1} \geq \binom{n}{k}$ if and only if $2k \leq n - 1$.

Roughly speaking, the sequence grows until the middle and then starts to decrease. Symmetry implies that if it grows to the middle then it must decay after the middle. The maximum is attained at $k = \lfloor n/2 \rfloor$.

Proof.

$$\frac{\binom{n}{k+1}}{\binom{n}{k}} = \frac{n!}{(k+1)!(n-k-1)!} \cdot \frac{k!(n-k)!}{n!} = \frac{n-k}{k+1}.$$

$\square$

Identities

We shall see several identities and prove them in various ways.

Claim 6. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Binomial formula.

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}.$$

$\square$

Combinatorial story. The number $\sum_{k=0}^{n} \binom{n}{k}$ is the total number of subsets of $[n]$, which is 2^n. $\square$

Claim 7. $\sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1}$.

Computation. Notice

$$k \binom{n}{k} = k \frac{n!}{k!(n-k)!} = n \frac{(n-1)!}{(k-1)!(n-k)!} = n \binom{n-1}{k-1}.$$

Thus,

$$\sum_{k=0}^{n} k \binom{n}{k} = \sum_{k=0}^{n} n \binom{n-1}{k-1} = n2^{n-1}$$

(need to make sure that $k = 0$ is also correct). $\square$

Analytic. By the binomial formula,

$$f(x) = (1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Differentiate

$$f'(x) = n(1 + x)^{n-1} = \sum_{k=0}^{n} k \binom{n}{k} x^{k-1}.$$

Substitute $x = 1$. $\square$

Combinatorial story. The number $\sum_{k=0}^{n} k \binom{n}{k}$ is the number of choices in: choose a group inside $[n]$ and then choose a leader for the group. We can first choose the leader and then choose the rest of the group inside a set of size $n - 1$. $\square$

Claim 8. *For $n > 0$, we have $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.*

Proof. Expand $0^n = (1 - 1)^n$ according to binomial identity. $\square$

The claim can be interpreted as follows:

$$\sum_{\text{even } k} \binom{n}{k} = \sum_{\text{odd } k} \binom{n}{k}.$$

In other words, the number of subsets of even size is equal to the number of subsets of odd size in $[n]$. For example, for $n = 4$, we have $1 + 6 + 1 = 4 + 4$. Try to find a one-to-one and onto map between the sets of even size to the sets of odd size.

Another way to write the claim is as follows.

Claim 9. *Let A be a finite non-empty set. Then,*

$$\sum_{S \subseteq A} (-1)^{|S|} = 0.$$

Estimations

Although $\binom{n}{k}$ is relatively simple to calculate, we sometimes need an estimate which is even easier to understand (this is a common theme in mathematics).

Theorem 10. $(n/k)^k \leq \binom{n}{k} \leq (ne/k)^k$ *for $0 < k \leq n$.*

Proof.

$$\binom{n}{k} \cdot (k/n)^k \leq \sum_{\ell=0}^{n} \binom{n}{\ell} (k/n)^\ell \qquad \text{(all terms are positive)}$$
$$= (1 + (k/n))^n$$
$$\leq e^k. \qquad (1 + x \leq e^x \text{ for all } x)$$

$\square$

If $k \ll n$, we can usually think of $\binom{n}{k}$ as n^k. If $k \approx n$, we can think of $\binom{n}{k}$ as c^n for some $c > 1$.

Theorem 11. $\binom{2n}{n} \geq 2^n/(n+1)$.

Proof. Unimodality implies $\binom{2n}{k} \leq \binom{2n}{n}$. Thus,

$$2^n = \sum_{k=0}^{2n} \binom{2n}{k} \leq (n+1) \binom{2n}{n}.$$

$\square$

In fact, $\binom{2n}{n}$ is, up to constants, equal to $2^{2n}/\sqrt{n}$. This can be proved using Stirling's formula $(n! \approx \sqrt{n}(n/e)^n)$.

3.4 Catalan numbers

Assume that we have n brackets. In how many ways can we legally[1] position them?
   Examples: for n equals

1. we have 1 option: ().

2. we have 2 options: ()(), (()).

3. we have 5 options: ()()(), (())(), (()()), ((())), ()(()).

4. we have 14 options.

5. we have 42 options.

6. we have 132 options.

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[1]A legal positioning of brackets is of the form (()(())).

Perhaps the simplest way to understand them is write a recursive formula. The observation is that the first bracket is closed somewhere $E = (E_1)E_2$ where E_1, E_2 are legal positioning of brackets with E_1 have k brackets and E_2 having $n - k - 1$. So, if we denote by $C(n)$ the number we are interested in then

$$C(0) = C(1) = 1$$

and for $n > 1$ we have

$$C(n) = \sum_{k=0}^{n-1} C(k)C(n - 1 - k).$$

For example,

$$C(2) = C(0)C(1) + C(1)C(0) = 2$$

and

$$C(3) = C(0)C(2) + C(1)C(1) + C(2)C(0) = 2 + 1 + 2 = 5.$$

We can thus calculate $C(n)$ exactly, but this may take a lot of time.

We now find a "formula" for $C(n)$. The basic idea is to "encode" a legal positioning of brackets as a walk on a grid with certain properties, and then count the number of walks.

We consider walks on the lattice of points in the plane of the form (x, y) where x, y are integers and $x + y$ is even. Draw it.

Our walks will always start at $(0, 0)$, and in each time step $t \geq 0$ we move from (x_t, y_t) to one of two positions $(x_t + 1, y_t + 1)$ or $(x_t + 1, y_t - 1)$. Draw the two options.

We interpret a sequence of brackets: (means $+1$ and) means -1. For example (()) gives $(0, 0), (1, 1), (2, 2), (3, 1), (4, 0)$.

A legal sequence of n brackets gives a walk from $(0, 0)$ to $(2n, 0)$ that stays above the line $y = 0$, and vice versa. So $C(n)$ is the number of such walks.

It is in fact easier to calculate a different quantity first. Denote by $W = W(n)$ the total number of walks from $(0, 0)$ to $(2n, 0)$. (Thus, $C(n) < W(n)$.) Every such walk has n times with a "$+1$ step" and n times with a "-1 step", so

$$W(n) = \binom{2n}{n}.$$

Denote by $\bar{C}(n)$ the number of walks that are *not* of "Catalan" form; walks $z = ((x_0, y_0), \ldots, (x_{2n}, y_{2n}))$ with $y_{2n} = 0$ that hit the line $y = -1$. That is, there exists $t \geq 0$ so that $y_t = -1$. Thus,

$$C(n) = W(n) - \bar{C}(n).$$

We first compute $\bar{C}(n)$. Given a walk $z = ((x_0, y_0), \ldots, (x_{2n}, y_{2n}))$ that contributes

to $\bar{C}(n)$, let
$$T(z) = \min\{t \geq 1 : y_t = -1\};$$

the first time the walk hits the line $y = -1$. Thus, $T(z) < 2n$.

For such z, define $R(z)$ as the walk obtained from z by reflection after time $t(z)$ with respect to the line $y = -1$. Draw an example. That is, for $t \leq T(z)$ we have $(R(z))_t = z_t$ and for $t > T(z)$ we have $(R(z))_t = (t, -y_t - 2)$. Thus, for all such z, we have $(R(z))_{2n} = (2n, -2)$.

We claim that the map $z \mapsto R(z)$ is one-to-one and onto the set of walks from $(0,0)$ to $(2n, -2)$. Indeed, this holds since the map R is invertible (it is its own inverse) and since every walk from $(0,0)$ to $(2n, -2)$ crosses the line $y = -1$ at least once.

To compute the number of walk from $(0,0)$ to $(2n, -2)$, notice that every such walk has $n - 1$ times with a "+1 step" and $n + 1$ times with a "−1 step". The number of such walks is therefore
$$\bar{C}(n) = \binom{2n}{n+1} = \binom{2n}{n-1}.$$

Finally,
$$C(n) = \binom{2n}{n} - \binom{2n}{n+1} = \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n+1)!(n-1)!} = \frac{(2n)!}{n!n!}\left(1 - \frac{n}{n+1}\right) = \frac{1}{n+1}\binom{2n}{n}.$$

As an exercise, you can verify that it satisfies the recursion we saw before. We may conclude that
$$C(n) \approx 2^{2n}/n^{3/2}.$$

Exercise 12. *Prove that $C(n)$ is equal to the number of triangulations of a convex $(n+2)$-gon in the plane.*

Examples: $C(1)$ corresponds to the triangle, $C(2)$ corresponds to the 2 triangulations of a square, etc.

3.5   Inclusion-exclusion

Assume that we have n finite sets $A_1, \ldots, A_n$, and we know their sizes. What can we say about the size of their union $\bigcup_{i=1}^n A_i$?

Theorem 13. $|\bigcup_{i=1}^n A_i| \leq \sum_{i=1}^n |A_i|.$

Proof. Let
$$A = \bigcup_i A_i.$$

For $a \in A$, let
$$I_a = \{i \in [n] : a \in A_i\}.$$

Thus, $|I_a| \geq 1$ for all $a \in A$. So,

$$|A| = \sum_{a \in A} 1 \leq \sum_{a \in A} |I_a| = \sum_{a \in A} \sum_{i=1}^{n} \mathbf{1}_{a \in A_i} = \sum_{i=1}^{n} \sum_{a \in A} \mathbf{1}_{a \in A_i} = \sum_{i=1}^{n} |A_i|.$$

$\square$

This inequality is far from being tight, for example if $A_1 = A_2 = \cdots = A_n$. So, if we have more data we can say more accurate things. If for example we know the sizes of the pairwise intersections then we can bound:

Theorem 14. $|\bigcup_{i=1}^{n} A_i| \geq \sum_{i=1}^{n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j|.$

Left as exercise. $\square$

If we have information on all intersection sizes, we can compute the size exactly. Example: If X, Y, Z are finite sets then

$$|X \cup Y \cup Z| = |X| + |Y| + |Z| - |X \cap Y| - |X \cap Z| - |Y \cap Z| + |X \cap Y \cap Z|.$$

Draw a Venn diagram. In general:

Theorem 15 (Inclusion-exclusion)**.**

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| +$$
$$+ \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| + \ldots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$
$$= \sum_{S \subseteq [n] : S \neq \emptyset} (-1)^{|S|+1} \left| \bigcap_{i \in S} A_i \right|.$$

Proof. Recall that for every finite non-empty B, we have $\sum_{S \subseteq B} (-1)^{|S|} = 0$ so we have a complicated way to write one:

$$1 = \sum_{S \subseteq B : S \neq \emptyset} (-1)^{|S|+1}.$$

Let
$$A = \bigcup_{i=1}^{n} A_i.$$

For $a \in A$, let
$$I_a = \{i \in [n] : a \in A_i\}.$$

For every $a \in A$ and non-empty $S \subseteq [n]$, we have that $S \subseteq I_a$ if and only if $a \in \bigcap_{i \in S} A_i$.

Therefore,

$$
\begin{aligned}
|A| &= \sum_{a \in A} 1 \\
&= \sum_{a \in A} \sum_{S \subseteq I_a : S \neq \emptyset} (-1)^{|S|+1} \\
&= \sum_{S \subseteq [n] : S \neq \emptyset} (-1)^{|S|+1} \sum_{a \in A : S \subseteq I_a} 1 \\
&= \sum_{S \subseteq [n] : S \neq \emptyset} (-1)^{|S|+1} \left| \bigcap_{i \in S} A_i \right|.
\end{aligned}
$$

$\square$

3.6 Fibonacci numbers

Leonardo Pisano Bigollo studied the population size of rabbits in the 12th century in Pisa (as his name suggests). His name later became Fibonacci (some explanation to name change: "filius Bonacci" used in the title of his book which means "the son of Bonaccio"). He came up with the following sequence: Think of $F(n)$ the population size in generation n. Assume that there is one female rabbit at time 1, that rabbits live forever, and that at each month from the second month it is born, every female produces one extra female. How many female rabbits are there after n months?

That is, what is $F(n)$ the is defined as follows? $F(1) = 1$, $F(2) = 1$, $F(3) = 2$, and in general for $n > 1$, all the rabbits from time $n - 1$ are alive at time n, and all rabbits from time $n - 2$ reproduce, so

$$F(n) = F(n-1) + F(n-2).$$

We have a recursive formula for $F(n)$, but can we find a formula that gives more information about $F(n)$? The answer is yes, and we will use simple linear algebra to do so.

For $n \geq 0$, define a two-dimensional vector

$$v_n = (F(n), F(n+1)).$$

So $v_0 = (0, 1)$, $v_1 = (1, 1)$ and so forth (we set $F(0) = 0$). The first observation is that the recursive definition of $F(n)$ implies that

$$v_{n+1} = Av_n$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

In particular, we get the following simple formula:

$$v_n = A^n v_0.$$

But what is A^n? To understand A^n better, we diagonalize A. Write

$$A = U^{-1}DU,$$

where

$$U = \begin{bmatrix} \frac{\sqrt{5}-1}{2} & 1 \\ \frac{-\sqrt{5}-1}{2} & 1 \end{bmatrix}$$

is orthogonal $UU^T = I$ and

$$D = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}$$

is diagonal. We will not explain how to find U, D for now. But how is this helpful? Well,

$$A^n = (UDU^{-1})(UDU^{-1}) \cdots (UDU^{-1}) = UD^nU^{-1} = U \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix} U^{-1}.$$

Since $v_n = A^n v_0$, we conclude that there are two real numbers a, b so that for all n,

$$F(n) = a \left(\frac{1 + \sqrt{5}}{2}\right)^n + b \left(\frac{1 - \sqrt{5}}{2}\right)^n.$$

We can find a, b using $F(0) = 0$ and $F(1) = 1$. These two equalities imply $a = -b$ and

$$1 = a \left(\frac{1 + \sqrt{5}}{2}\right) - a \left(\frac{1 - \sqrt{5}}{2}\right)$$

or

$$a = \frac{1}{\sqrt{5}}.$$

Finally, for all n,

$$F(n) = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n.$$

It may be surprising that the r.h.s. above is even integer. It also tells us that for large n, the value of $F(n)$ is close to $\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n$.

The golden ratio

The number $\phi = \frac{1+\sqrt{5}}{2}$ is called the golden ratio. It is one of the solutions to the equation $x^2 - x - 1 = 0$, or $x = 1/(x-1)$. A $1 \times \phi$ rectangles thus have some symmetry (draw). Another equation is $x = 1 + 1/x$, which shows that as infinite fraction ("shever meshulav") it is equal to

$$\phi = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\cdots}}}.$$

The sequence

$$\frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \dots$$

converges to ϕ.

This is related to that the Fibonacci sequence appears in many spirals in nature, like in pine cones where the number of spirals of "opposite" type are two consecutive numbers in sequence.

3.7   Linear recurrences

The Fibonacci sequence is just one example of a more general phenomenon.

Definition 16. *A sequence $F(1), F(2), \dots$ is a (homogenous) linear recurrence of order r if for every $n > r$ we have*

$$F(n) = a_1 F(n-1) + a_2 F(n-2) + \dots + a_r F(n-r)$$

where $a_1, \dots, a_r$ are some constants.

The Fibonacci sequence is of order two. Given $a_1, \dots, a_r$, to specify F we need to choose $F(1), \dots, F(r)$.

The collection of sequences satisfying the rule is a vector space of dimension r. The collection of Fibonacci sequences is a vector space of dimension two.

We now explain how to find a formula for the general term of such a sequence. We can use linear algebra, as we did for the Fibonacci sequence, but here we use a seemingly

different approach (which is actually equivalent). We start by searching for sequences of the form

$$F(n) = x^n$$

for some $x \in \mathbb{R}$. This gives for every $n > r$ the equation

$$x^n = a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_r x^{n-r}.$$

The value $x = 0$ is always a solution, and if $x \neq 0$ this is equivalent to

$$p(x) = x^r - a_1 x^{r-1} - a_2 x^{r-2} - \ldots - a_r = 0,$$

which is a polynomial equation of degree r.

Definition 17. *The polynomial $p(x)$ is the characteristic polynomial of the recurrence formula.*

For Fibonacci,
$$p(x) = x^2 - x - 1.$$

For now, let us assume that this equation has r different real solutions (there are always at most r solutions, but sometimes less than r, and sometimes some roots are the same). For Fibonacci the 2 roots are

$$x_1 = \frac{1 + \sqrt{5}}{2}, \quad x_2 = \frac{1 - \sqrt{5}}{2}.$$

So the r sequences
$$(x_1^n), (x_2^n), \ldots, (x_r^n)$$

satisfy the recurrence formula. In fact, they form a basis to the space of sequences. Specifically, we can write every F as

$$F(n) = \alpha_1 x_1^n + \alpha_2 x_2^n + \ldots + \alpha_r x_r^n.$$

How do we find $\alpha = (\alpha_1, \ldots, \alpha_r)$? Using the initial conditions: We have r equations

$$F(i) = \alpha_1 x_1^i + \alpha_2 x_2^i + \ldots + \alpha_r x_r^i$$

for $1 \leq i \leq r$, and we have r variable, α.

For Fibonacci,

$$0 = F(0) = \alpha_1 x_1^0 + \alpha_2 x_2^0, \quad 1 = F(1) = \alpha_1 x_1^1 + \alpha_2 x_2^1,$$

which yields

$$\alpha_1 = \frac{1}{\sqrt{5}}, \ \alpha_2 = -\alpha_1.$$

In general, there is always a solution if the r roots are different. This is a theorem from linear algebra, which we shall not prove.[2]

Let us summarize.

1. We are given a recurrence formula of order r, as above. That is, we are given $a_1, \ldots, a_r$ and $F(1), \ldots, F(r)$.

2. This defines the characteristic polynomial $p(x)$, which has degree r.

3. We try to find all root of p.

4. If p has r different roots, then the formula of a generic term in sequence is of the form $F(n) = \sum_{i=1}^{r} \alpha_i x_i^n$.

5. To find α, we need to solve r equations in r variables.

Comments:

1. You will see examples in the exercise.

2. Similar to differential equations.

We did not cover:

What if not all roots are distinct? What if the formula is not homogenous? ...

3.8   Generating functions

Generating functions is a way to use abstract algebra to solve combinatorial problems (there are applications in other areas as well).

First, the definition:

---

[2]Why? We can write it in matrix form:

$$f = X\alpha$$

where $f = (F(1), \ldots, F(r))$, and $X_{i,j} = X_j^i$ is $r \times r$. If all roots are equal there is a unique solution.

Theorem 18. *If $x_1, \ldots, x_r$ are r distinct numbers, then the matrix X is invertible.*

Hint. $\det(X) \neq 0$.   □

Definition 19. *A generating function f is of the form*

$$f = \sum_{n=0}^{\infty} f_n x^n.$$

It is a formal infinite sum.

Arithmetic

We can sum and multiply two functions f, g. The rules are:

$$(f + g)_n = f_n + g_n$$

and

$$(f \cdot g)_n = \sum_{k=0}^{n} f_k g_{n-k}.$$

These operations are commutative.

Examples:

The function $0 := 0 + 0 \cdot x + 0 \cdot x^2 + \ldots$ is so that $f + 0 = f$ for all f, and the function $1 := 1 + 0 \cdot x + 0 \cdot x^2 + \ldots$ is so that $1 \cdot f = f$ for all f.

Comment:

We do not care for now if f actually defines a function, but think of it abstractly.

One more example:

The function $f = \sum_{n=0}^{\infty} x^n$, or $f_n = 1$ for all n. We claim that the function $g = 1 - x$, for which $g_0 = 1, g_1 = -1$ and $g_n = 0$ for $n > 1$, is the inverse of f. That is, $g \cdot f = 1$. Indeed, for $n > 1$,

$$(g \cdot f)_n = \sum_{k=0}^{n} g_k f_{n-k} = f_n - f_{n-1} = 0$$

and

$$(g \cdot f)_0 = g_0 f_0 = 1.$$

We write it as

$$(1 - x)f = 0$$

or

$$f = \frac{1}{1 - x}.$$

Fibonacci

We now see now this algebraic object can help in finding a formula for the Fibonacci numbers. Let $f_0 = 0, f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n > 2$. This is the generating function of the Fibonacci sequence.

We translate the definition of f to a functional equation it must satisfy:

Claim 20. $f = xf + x^2 f + x$.

Stated differently:

$$f = \frac{x}{1 - x - x^2}.$$

Proof. First, what is xf? Well, for $n = 0$, we have $(xf)_0 = 0$ and for $n > 0$ we have

$$(xf)_n = f_{n-1}.$$

Similarly, $(x^2 f)_0 = (x^2 f)_1 = 0$ and for $n > 1$,

$$(x^2 f)_n = f_{n-2}.$$

Write it as $01123\ldots, 00112\ldots, 00011\ldots$ to see that x is "missing."   □

To understand $\frac{1}{1-x-x^2}$ better, first write $1 - x - x^2 = (x - \phi_1)(x - \phi_2)$ where ϕ_1, ϕ_2 are the 2 roots of the equation $1 - x - x^2 = 0$ (of course related to golden ratio). Now, search for a_1, a_2 so that

$$\frac{1}{1 - x - x^2} = \frac{a_1}{x - \phi_1} + \frac{a_2}{x - \phi_2}.$$

How do we find a_1, a_2? Sum the r.h.s. and compare coefficients. We are almost done, since

$$\frac{a_1}{x - \phi_1} = -\frac{a_1}{\phi_1}\frac{1}{1 - (x/\phi_1)} = -\frac{a_1}{\phi_1}\sum_{n=0}^{\infty}(x/\phi_1)^n = -\frac{a_1}{\phi_1}\sum_{n=0}^{\infty}(1/\phi_1)^n x^n.$$

Similarly for the other term, and overall we found an explicit formula for f_n. Try to fill in the details by yourself.

Local summary

Sequence $(F(n)) \to$ generating function $f \to$ functional equation $E(f) = 0 \to$ explicit description of F.

Catalan numbers

We now demonstrate a similar process for the Catalan numbers. Define the generating function c by $c_n = C_n$. The functional equation:

Exercise 21. $xc^2 - c + 1 = 0$.

Solve for c:

$$c = \frac{1 \pm \sqrt{1 - 4x}}{2x}.$$

Only one of the solution is a legal generating function (the one with the minus sign—why?). To understand the formula of the Catalan numbers, we need to understand the function $\sqrt{1 - x}$.

Solution. We have seen

$$C_0 = C_1 = 1$$

and for $n > 1$,

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}.$$

Write

$$(c^2)_n = \sum_{k=0}^{n} c_k c_{n-k} = c_{n+1}$$

or

$$c^2 = \sum_{n=0}^{\infty} c_{n+1} x^n.$$

Stated differently,

$$xc^2 = x \sum_{n=0}^{\infty} c_{n+1} x^n = \sum_{n=0}^{\infty} c_{n+1} x^{n+1} = c - 1,$$

or

$$xc^2 - c + 1 = 0.$$

□

Summary

We have seen several basic problems and tools in discrete mathematics, like inclusion-exclusion, using linear algebra, using abstract algebra, and using reflection (geometry).

Chapter 4

Existential proofs: the pigeonhole principle

A typical mathematical statement is of the form "object X exists". Some examples: a function has a root, every injective map from $\mathbb{N}$ to $\mathbb{R}$ is not onto, and so forth. We shall see several basic methods for proving such a statement.

The pigeonhole principle (a.k.a. Dirichlet principle) states (roughly speaking) that if $n + 1$ pigeons fly to n holes then there is at least one hole with at least two pigeons. More formally, there is no bijection between $[n+1]$ and $[n]$. It is a very simple principle but it leads to non trivial statements. We shall see several examples.

Numbers

Claim 22. *Let $A \subseteq \{0, 1, \ldots, 9\}$ be a set of size 6. There are two numbers $a_1 \neq a_2$ in A that sum to 9.*

Proof. Holes: $\{0, 9\}, \{1, 8\}, \{2, 7\}, \{3, 6\}, \{4, 5\}$.

Pigeons: A.

The principle tells us that 2 elements of A are in same hole ($|A| = 6$ and number of holes is 5). $\qquad \square$

In this case the choice of pigeons and holes is clear. In other cases it may not be as clear.

Geometry

Claim 23. *If we throw 5 darts into a target that is an equilateral triangle with side length 1, then there are 2 darts of distance at most $1/2$.*

Proof. Pigeons: the darts.

Holes: Partition the target to 4 equilateral triangles of side length $1/2$.

The principle says that 2 darts fall in the same small triangle, whose diameter is $1/2$. □

If there are 4 darts, the conclusion may not hold (what is the picture?). The $1/2$ is tight even for 6 darts (what is the picture?).

Sequences

We now describe a more technical application.

Definition 24. *A sequence $a_1, \ldots, a_n$ is increasing if $a_{i+1} \geq a_i$, and is decreasing if $a_{i+1} \leq a_i$ for all $i = 1, \ldots, n-1$. A subsequence of length k is determined by $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ and is of the form $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$.*

Example:

$1, 2, 5, 3, 4$ is not monotone. $1, 2, 4$ is a increasing sub-sequence. $5, 4$ is a decreasing sub-sequences.

Theorem 25 (Erdos-Szekeres)**.** *Every sequence $a_1, \ldots, a_m$ with $m = n^2 + 1$ has a mono-tone sub-sequence of length at least $n + 1$.*

Proof. Pigeons: The elements of sequence. There are $m = n^2 + 1$ of them.

Holes: This time is not as clear. The holes are elements of $[n] \times [n]$. There are n^2 holes.

How do we map pigeons to holes? Given $i \in [m]$, define two numbers p_i and q_i and follows. p_i is the length of the longest increasing sub-sequence starting at a_i, and q_i is the length of the longest decreasing sub-sequence starting at a_i.

If the conclusion of the theorem does not hold, then the map $i \mapsto (p_i, q_i)$ takes values in the holes $[n] \times [n]$.

Assume towards a contradiction that $p_i = p_j$ and $q_i = q_j$ for $i < j$. Consider

$$a_1, \ldots, a_i, a_{i+1}, \ldots, a_{j-1}, a_j, \ldots, a_s.$$

Observe that if $a_i > a_j$ then $q_j < q_i$ since every decreasing sub-sequence from a_j can be extended by a_i. Similarly, if $a_i < a_j$ then $p_i > q_j$. A contradiction. □

Diophantine approximations

Diophantine approximations are about approximating real numbers by rational. There is a lot of theory behind it, and we will see a simple example.

Every real number x can be approximated by a rational p/q up to accuracy $1/q$. Dirichlet's approximation theorem tells us that we can do better.

Theorem 26. *Let $x \in [0,1]$ and $n \in \mathbb{N}$. There are integers $0 \leq p \leq q \leq n$ so that*

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{nq}.$$

Proof. Partition the interval $[0,1]$ to n equal length interval (the holes). Consider the $n+1$ numbers $a_0 = 0 \cdot x, a_1 = 1 \cdot x, a_2 = 2 \cdot x, \ldots, a_n = n \cdot x$ modulo $\mathbb{N}$. These numbers are in $[0,1)$. By the pigeon hold principle two of them fall in same interval: there are $k_1 < k_2$ so that $|a_{k_2} - a_{k_1}| \leq 1/n$ or

$$k_2 x + n_2 - (k_1 x + n_1) \in [0, 1/n].$$

Thus,

$$|(k_2 - k_1)x - (n_1 - n_2)| \leq 1/n.$$

$\square$

Averages

We now discuss a generalization of the pigeon hold principle. It states e.g. that in a given population at least one person is as high as the average height.

Theorem 27. *If $x_1, \ldots, x_n$ are real numbers so that $\sum_{i=1}^{n} x_i = S$ then there is $i \in [n]$ so that $x_i \geq S/n$.*

Proof. Assume towards a contradiction that $x_i < S/n$ for all i. Sum these to gety

$$\sum_{i=1}^{n} x_i < \sum_{i=1}^{n} S/n = S,$$

a contradiction. $\square$

We will not see applications of this theorem right now, but later on we shall see it is surprisingly powerful.

Chapter 5

Graphs

The graphs we talk about (not of a function) are a collection of points with edges connecting them. Draw an example.

Definition 28. *A (simple and undirected) graph G is a pair (V, E) so that V is a finite set and E is a subset of $\binom{V}{2} = \{\{u, v\} : u \neq v \in V\}$.*

Graphs model many natural objects, like a society and "friendship" or a net of computers or cities and roads.

Examples:

K_n, cycle of length n, path of length n.

5.1   Basic notions

Connectivity

A path of length m in a graph is a sequence of vertices $v_1, v_2, \ldots, v_m \in V$ so that v_i, v_{i+1} is connected by an edge for all $i < m$. Draw an example. A path is called simple if no vertex appears more than once on it. A cycle is a path so that $v_1 = v_m$.

The connected component of a vertex v is the set of all $u \in V$ so that there is a path from v to u. It is the set of vertices that can be reached from v, if edges are roads.

The graph G can be partition to connected components. In fact, connectedness defines an equivalence relation on V. If G has one connected component, we say it is connected. Connectedness is a topological property.

Degrees

The degree $deg(v)$ or d_v of a vertex v is the number of edges it belongs to:

$$deg(v) = |\{e \in E : v \in e\}|.$$

Examples.

Claim 29. *For every G, there are $u \neq v$ in $V(G)$ so that $deg(u) = deg(v)$.*

Proof. Let $n = |V(G)|$. Use the pigeon hole principle. The n pigeons are the vertices. The degree of v is between 0 and $n-1$. There are n options. So, we can not use principle as is. There are 2 cases:

   If there is a vertex with degree $n-1$, then all degrees are at least 1, and the number of holes is $n-1$.

   Otherwise, the number of holes is also $n-1$: the integers between 0 and $n-2$.   □

Claim 30. *For every G, the sum $\sum_v deg(v)$ is even.*

   Example.

Proof. For $v \in V$ and $e \in E$, let $f(v, e) = 1$ if $v \in e$ and 0 othewise. Thus,

$$\sum_v deg(v) = \sum_v \sum_e f(v, e) = \sum_e \sum_v f(v, e) = \sum_e 2 = 2|E|.$$

□

Distances

A graph also defines a metric space. The points in the space are vertices. The distance between $v, u \in V$, denoted $dist(v, u)$, is the length of the shortest path connecting v and u, and is ∞ if they are not connected. Recall that a function $\rho : V \times V \to \mathbb{R}$ is a metric if

1. $\rho(v, u) \geq 0$ for all u, v.

2. $\rho(v, u) = 0$ iff $u = v$.

3. $\rho(v, u) = \rho(u, v)$ for all u, v.

4. $\rho(v, u) \leq \rho(v, w) + \rho(w, u)$ for all u, v, w.

The last property is called the triangle inequality.

Claim 31. *The function $dist(\cdot, \cdot)$ is a metric.*

Proof. We just prove the triangle inequality. Let u, v, w so that $dist(u, w), dist(w, v) < \infty$. (If one of the distance is infinite, the inequality holds.) Let $v_1, \ldots, v_m$ be a shortest path from u to w, and let $v'_1, \ldots, v'_{m'}$ be the shortest path from w to v. Thus, $v_1, \ldots, v_m, v'_1, \ldots, v'_{m'}$ is a path from u to v. □

As usual, the metric structure contains more data than the topological structure.

Claim 32. *v, u are in same connected component iff $dist(v, u) < \infty$.*

Balls

A ball of radius r around a vertex v is

$$B(v, r) = \{u : dist(u, v) \leq r\}.$$

A sphere of radius r consists of all vertices of distance exactly r. A ball is a disjoint union of spheres.

Diameter

The diameter of a graph is the maximum distance between two vertices in it. Draw a shape in plane and see its diameter.

Geodesics

In plane, between every two points A and B there is a single curve which is of shortest length - the straight line. It corresponds to the "best way" to get from A to B.

In a graph, there is a similar notion. A geodesic between u, v is a shortest path between them. It is, however, not necessarily unique. E.g. the $n \times n$ grid.

Nets

A ϵ-net in the unit circle is a set of points that approximates every point in the circle up to ϵ. It is helpful to find ϵ-nets that are as small as possible.

Definition 33. *An r-net in a graph is a set of vertices U so that*

- *For every $u \neq u'$ in U, we have $dist(u, u') > r$.*

- *For every v in V, there is u in U so that $dist(v, u) \leq r$.*

Claim 34. *For every r, in every graph there is an r-net.*

This is a "standard proof" in say compact domains.

Proof. Let U be a maximal set so that for every $u \neq u'$, we have $dist(u, u') > r$. Clearly, it satisfies the first property. The second property holds since if there is v in V so that $dist(v, u) > r$ for all u in U, then we can add v to U, which is a contradiction to its maximality. $\square$

5.2   Trees

A tree is a connected graph with no (simple) cycles. That is, between every two vertices u, v there is a unique path connecting them. Trees have to balance between two opposite forces: For the graph to be connected, we need many edges. For the graph to have no cycles, we need few edges. We study this tension formally.

Claim 35 (Connected graphs have many edges). *If G is connected then $|E| \geq |V| - 1$.*

Proof. Proof by induction on $n = |V|$. For $n = 1$, this holds. Assume this holds for $n-1$. Let G be a graph with n vertices. Let v be a vertex of G. Let G' be the graph obtained from G by deleting v and the edges connected to it. The graph G' is not necessarily connected. Let $G'_1, \ldots, G'_k$ be the connected components of G'. By induction,

$$|E(G'_i)| \geq |V(G'_i)| - 1$$

and we can sum

$$\sum_{i=1}^{k} |E(G'_i)| \geq -k + \sum_{i=1}^{k} |V(G'_i)|.$$

Note that

$$|V(G)| = 1 + \sum_{i=1}^{k} |V(G'_i)|.$$

Since G is connected, for every $i \in [k]$, there is v'_i in $V(G'_i)$ so that $\{v, v'_i\} \in E(G)$. Thus,

$$|E(G)| \geq k + \sum_{i=1}^{k} |E(G'_i)|.$$

$\square$

Claim 36 (Acyclic graph have few edges). *If G is acyclic then $|E| \leq |V| - 1$.*

Proof. Again, by induction on $n = |V|$. For $n = 1$, this is true. Assume this is true for $n - 1$. Let G be a graph with $|V| = n$. Let $v \in V(G)$ and let $v_1, \ldots, v_k$ be the vertices so that $\{v, v_i\} \in E(G)$ for all i. Let G' be the graph obtained from G by deleting v and its edges. Let $G'_1, \ldots, G'_m$ be the connected components of G'. The graph G' is acyclic.

The observation is that for every $i \in [m]$, there is at most one v_j that belongs to G'_i, since otherwise we have a cycle in G. So, $m \geq k$ and

$$|E(G)| = k + \sum_{i=1}^{m} |E(G'_i)| \leq k + \sum_{i=1}^{m} (|V(G'_i)| - 1) \leq k - m + |V| - 1.$$

$\square$

Corollary 37. *If G is a tree then $|E(G)| = |V(G)| - 1$.*

Corollary 38. *If G is a tree with more than one vertex then there is $v \in V(G)$ so that $deg(v) = 1$.*

Proof. Since $G = (V, E)$ is connected, all degrees are positive. On the other hand,

$$\sum_{v} deg(v) = 2|E| = 2(|V| - 1)$$

or

$$\frac{1}{|V|} \sum_{v} deg(v) = \frac{2(|V| - 1)}{|V|} < 2.$$

By the extended pigeon hole principle, there is v so that $deg(v) < 2$. $\square$

The last corollary can be used to prove that all trees can be recursively constructed by adding one vertex at a time and connecting it to a smaller tree with one edge. Topologically, this means that trees are contractible.

Analogy

There is a similarity between the notion of "tree" and the notion of a "basis" in linear algebra. Connectivity corresponds to spanning. Cycles correspond to linear dependence. A tree is a connected a cyclic graph. A basis is both spanning and independent.

5.3 Bipartite graphs

A special type of graphs that appear in many natural contexts are bipartite graphs. A graph $G = (V, E)$ is bipartite is V is the disjoint union of V_0, V_1 and V_0, V_1 are independent sets (i.e., contain no edges). The sets V_0, V_1 are sometimes called color classes. They can be naturally drawn with two sides. The model things like: workers and tasks, or men and women.

The following theorem describes an important property of bipartite graphs.

Theorem 39. *Let G be a graph. Then G is bipartite iff all cycles in G are of even length.*

Proof. First, assume that G is bipartite. Consider a cycle $v_0, v_1, \ldots, v_m = v_1$ in G. Assume without loss of generality that $v_0 \in V_0$, then $v_1 \in V_1, v_2 \in V_0, v_3 \in V_1$ and so forth. Specifically, since $v_m = v_1 \in V_0$ we know that m is even.

Second, assume that G does not contain an odd cycle. Assume without loss of generality that G is connected (otherwise, perform the following for each connected component separately). What are the color classes of G? Fix $v \in V$. Denote by V_0 the set of vertices of even distance from v, and let V_1 be the complement of V_0. If V_0 is not independent then we found an odd cycle: If $\{u, w\} \in E$ with $u, w \in V_0$ then $v \to u \to w \to v$ is an odd cycle. Similarly, when $u, w \in V_1$.   □

5.4   Planarity

A graph G is planar if there is a drawing of it in the plane so that the vertices are points, the edges are continuous paths, and two disjoint edges do not cross each other.

Some graphs are planar and some are not (it is a "topological" property).

Theorem 40. $K_{3,3}$ *is not planar.*

How can we prove such a statement? We need to exclude all embeddings of the graph in plane. How can we argue on all embeddings? The short answer is that we find an invariant. A quantity which is X for all planar graph, but which is not X for $K_{3,3}$. Such a quantity was defined by Euler.

Consider a planar drawing of a graph G. The notion of a face appears only when G is drawn in the plane.

Definition 41. *A face is a connected component of $\mathbb{R}^2$ when deleting the drawing of the graph from the plane.*[1]

It is clear from the drawing what are the faces (we shall not be too formal about these issues). There is also a single infinite face. Denote by F the set of faces of the drawing.

Theorem 42 (Euler)**.** *If G is planar and connected then $|V| - |E| + |F| = 2$.*

Draw an example.

---

[1]A drawing is defined by maps $\delta_V : V \to \mathbb{R}^2$ $\delta_E : E \to$ continuous paths in plane so that δ_E respects δ_V and if $e \cap e' = \emptyset$ in E then $\delta_E(e) \cap \delta_E(e') = \emptyset$. A face is a connected component of $\mathbb{R}^2 \setminus D$, where $D = \delta_v(V) \cup \delta_e(E)$.

Comments

1. The vertices are 0-dimensional, the edges 1-dimensional, and the faces 2-dimension. So the sign is (-1) to the dimension. The number 2 is the Euler characteristic. (There are more general topological statements.)

2. If G is not connected, the theorem holds for each component separately. The sum "encodes" the number of connected components. What is the theorem for general graphs? Hint: All of them share the same infinite face.

Proof. The proof is by induction in $|E|$. If the graph is a tree, then we known $|E| = |V| - 1$ and we can draw it with $|F| = 1$, so the theorem holds. If the graph contains a cycle C, consider the drawing of C in the place. C separates the plane to two parts. Let e be one of the edges of C. Let G' be the graph obtained from G by removing e. The edge e is on the boundary of two faces, and when we remove it, we merge these face to a single face. The graph G' is still connected, so by induction $|V'| - |E'| + |F'| = 2$. But $|V'| = |V|$, $|E'| = |E| - 1$ and $|F'| = |F| - 1$. □

The theorem implies for example that every planar graph has few edges.

Claim 43. *If G is connected planar and with at least three vertices then $|E| \le 3(|V|-2)$.*

Proof. If G has two edges, the theorem holds. Otherwise, consider a drawing of G in the plane. Observe that the number of edges that appear on the boundary ∂f of each face $f \in F$ is at least three. Also observe that each $e \in E$ is on the boundary of at most two faces. Therefore,
$$3|F| \le \sum_{e \in E} \sum_{f \in F} 1_{e \in \partial f} \le 2|E|,$$
and
$$|E| = |V| - 2 + |F| \le |V| - 2 + \frac{2}{3}|E|.$$
□

It is interesting to note that this upper bound is tight for triangulations: If G is a triangle then $|E| = 3(|V|-2)$. To make the graph larger, choose a triangle, add a vertex inside it, and connect it to the three vertices. This increased $|E|$ by three and $|V|$ by one.

We can now prove that some graphs are not planar. In general, a graph may have as many as $\binom{n}{2}$ edges (n is number of vertices). So for large n, we know that K_n is not planar since it has too many edges.

Corollary 44. *K_5 is not planar. Therefore, K_n is not planar for $n \ge 5$.*

Proof. If it was planar then

$$10 = \binom{5}{2} \leq 3(5-2) = 9.$$

□

This criterion does not suffice for proving that $K_{3,3}$ is not planar, since it has 6 vertices and 9 edges. But Euler's theorem does help to show that.

Theorem 45. $K_{3,3}$ *is not planar.*

Proof. Assume towards a contradiction that it is planar. Observe that $K_{3,3}$ does not contain any triangles. Therefore, every face in it has at least four edges on its boundary. So, similarly to before $4|F| \leq 2|E|$ and

$$9 = |E| = |V| - 2 + |F| \leq |V| - 2 + |E|/2 = 6 - 2 + 4.5 = 8.5.$$

□

We can also deduce the following general property.

Corollary 46. *In every planar graph there is a vertex of degree at most five.*

Proof. We may assume the graph is connected, and then

$$\frac{1}{|V|} \sum_{v \in V} deg(v) = \frac{2|E|}{|V|} \leq \frac{6(|V| - 2)}{|V|} < 6.$$

□

Platonic solids

Euler's theorem can be used to prove that there are only five platonic solids – these are 3-dimensional shapes (convex polyhedra) so that all faces have the same number of edges on their boundary, and all vertices have the same degree. See the following image. This is an amazing statement.

Tetrahedron Octahedron Cube

Icosahedron Dodecahedron
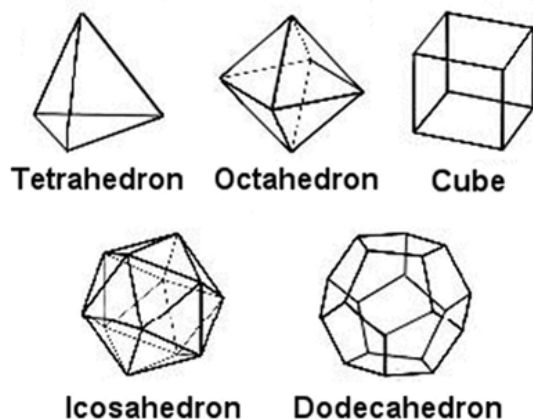
E.g. the cube has 8 vertices, 12 edges and 6 faces. There is a duality between these solids (switching vertices and faces). They can be partitioned to 2,2,1, such that each part consists of "duals" - who is its own dual? In the exercise, you will prove that these are the five options. The proof is by transforming such a solid into a planar graph by puncturing a face and stretching. After this transformation, using Euler's formula we get a relatively simple calculation.

5.5 Colorings

Consider a planar map (draw an example). What is the minimum number of colors needed to color the countries so that every two adjacent ones are colored differently?

From a planar map, we can define a dual graph. Its vertices are the faces of the map, and its edges are between two adjacent faces. This is part of "Poincare duality."

For general graphs:

Definition 47. *A k-coloring of G is a map $\phi : V \to [k]$ so that if $\{u, v\} \in E$ then $\phi(u) \neq \phi(v)$.*

Equivalently, for every $i \in [k]$, the set $\phi^{-1}(i)$ is independent in G.

Definition 48. *The chromatic number $\chi(G)$ is the minimum k for which there is a k-coloring of G.*

Draw some examples.

The question above stated differently: what is the maximum chromatic number of planar graphs?

One color

First, we consider the family of graphs G so that $\chi(G) = 1$. What is this family? The empty graphs.

Two colors

Second, we consider the family of graphs G so that $\chi(G) = 2$. What is this family?

Theorem 49. *The graph G has chromatic number two iff G is bipartite.*

Proof. If G has $\chi(G) = 2$ then $\phi^{-1}(1)$ and $\phi^{-1}(2)$ show that G is bipartite. If G is bipartite then $\phi(V_0) = 1$ and $\phi(V_1) = 2$ is a two-coloring of G.   □

Three colors

What is the family of 3-colorable graphs? This turns out to be a very difficult question. Although there is a simple criterion for deciding two-colorability (there is an efficient greedy algorithm), the problem of three-colorability is NP-complete, which makes it hard as far as we know (and most believe).

k-colors

Although, k-colorability is hard to decide, in some cases it is easy.

Theorem 50. *If the maximum degree of G is d then $\chi(G) \le d + 1$.*

Proof. The proof is by induction on $|V| = n$. For $n \le d + 1$ the claim holds. Assume $n > d + 1$. Let $v \in V(G)$ and $G' = G - v$. By induction $\chi(G') \le d + 1$. Extend the coloring ϕ' of G' to a coloring of G. Let $v_1, \ldots, v_t$ be the neighbors of v in G. By assumption $t \le d$. There is a value $i \in [d + 1]$ so that $i \notin \phi(\{v_1, \ldots, v_t\})$, and set $\phi(v) = i$ and $\phi(u) = \phi'(u)$ for $u \ne v$.   □

Planar graphs

Back to colorings of planar graphs. What is $\chi(G)$ if G is planar? It can be 2 - trees for example. It can be 3 - a triangle. It can be 4 - K_4 for example. Can it be five? It turns out that the answer is no. We shall not prove this, but let us start with:

Theorem 51. *If G is planar then $\chi(G) \le 6$.*

Proof. Use the following two properties:

  1. If G is planar then $G - v$ is planar too for every v.

  2. If G is planar there is a vertex in it with degree at most 5.

The previous proof by induction works.   □

Theorem 52. *If G is planar then $\chi(G) \le 5$.*

Proof. By induction on $n = |V|$. If $n \leq 5$ the claim holds. Assume $n > 5$. Let $v \in V(G)$ be of degree at most five, and let $G' = G - v$. The graph G' is planar and it has a 5-coloring ϕ' by induction. Let U be the set of neighbors of v in G. If $|\phi'(U)| < 5$ then we are done. Otherwise, $|U| = 5$ and ϕ' is a bijection on U. $U = \{u_1, \ldots, u_5\}$ is order in the plane counter-clock-wise with respect to v. Without loss of generality, assume $\phi'(u_i) = i$. (Draw it.)

We will show that we can change ϕ' to ϕ'' of G' so that $|\phi''(U)| < 5$. After this we are done, by the same reasoning as before.

Denote by G_1 the graph that G induces on $\phi^{-1}(\{1,3\})$. Denote by G_0 the graph that G induces on $\phi^{-1}(\{2,4\})$. There are three cases to consider:

- If u_1, u_3 are in different components in G_1. Let C_1 be the connected component of u_1 in G_1. Let ϕ'' be the same as ϕ', except that in C_1 it switches 1 and 3. By construction ϕ'' is a 5-coloring of G', with the required property ($\phi''(U) = \{2,3,4,5\}$).

- If u_2, u_4 are in the same components in G_0, a similar argument holds.

- We will show that it can not be that the two options above do not hold. Indeed, otherwise, there is a $1 - 3$ path from u_1 to u_3 and a $2 - 4$ path from u_2 to u_4. But this is topologically impossible in the plane.

$\square$

The only known proof of the following theorem uses a computer.

Theorem 53. *If G is planar then $\chi(G) \leq 4$.*

5.6 Ramsey numbers

Ramsey theory is about showing that "in every large system, there are ordered subsystem." We focus on the case of graphs (also historically first).

Definition 54. *The Ramsey number for $s, t \in \mathbb{N}$, denoted $R(s,t)$, is the minimum number n so that every graph G with n vertices either contains a clique[2] of size s or an independent set of size t.*

Claim 55. $R(2, n) = n$.

Proof. If G has n vertices: If there is an edge, we found a 2-clique. Otherwise, there are no edges.

An empty graph with $n - 1$ vertices, does not have this property. $\square$

---

[2]A subset of vertices all of which are connected.

When $s, t > 2$, it is harder.

Claim 56. $R(3, 3) = 6$. *In words, every graph with 6 vertices either contains a triangle or an independent set of size 3; and on the other hand, there is a graph with 5 vertices that avoids both.*

One part of this claim is: The cycle of length 5 has no clique or independent sets of size three.

Another part of this claim is: *In every room with 6 people, either at least 3 people know each other, or at least 3 people do not know each other ("knowing" is symmetric).*

Why? Consider person 1. If she knows at least 3 people there are 2 options: The 3 people do not know each other, and we are done. A pair of people she knows, knows each other, and we are done again. Otherwise, she does not know at least 3 people and the argument is similar. □

Here is a table with a few examples (due to symmetry we only list the numbers above the diagonal $t = s$):

s/t	2	3	4	5	6
2	2	3	4	5	6
3		6	9	14	18
4			18	25	$35 - 41$
5				$43 - 49$	$58 - 87$
6					$113 - 298$

A quote from Erdos: "Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for five and five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for six and six, however, we would have no choice but to launch a preemptive attack."

First, we prove these number are finite. We actually provide a pretty good upper bound.

Lemma 57. *For $s, t > 1$, the number $R(s, t)$ is finite. Moreover, if $s, t > 2$ we have*

$$R(s, t) \leq R(s - 1, t) + R(s, t - 1).$$

Proof. By induction. Base: $R(s, t) = t$ is finite. Step: Since $R(s - 1, t), R(s, t - 1)$ are finite by induction, let

$$n = R(s - 1, t) + R(s, t - 1).$$

Let G be a graph with n vertices. Let v be a vertex in G and $d = deg(v)$. There are two cases. If $d \geq R(s - 1, t)$ then in the set of v's neighbors we either found an

$s - 1$ clique or a t-independent-set. Otherwise, the number of non-neighbors is at least $n - 1 - d \geq n - 1 - (R(s - 1, t) - 1) = R(s, t - 1)$, and a similar argument works. $\square$

Corollary 58. $R(s, t) \leq \binom{s+t-2}{s-1}$.

Proof. By induction (say on $s + t$).
   Base: for $s = 2$ we have $R(2, t) = t \leq \binom{t}{1}$.
   Step: By above and induction,

$$R(s, t) \leq R(s - 1, t) + R(s, t - 1) \leq \binom{s - 1 + t - 2}{s - 2} + \binom{s + t - 1 - 2}{s - 1} = \binom{s + t - 2}{s - 1}.$$

$\square$

Of special interest are the diagonal numbers.

Corollary 59. $R(s, s) \leq \binom{2s-2}{s-1} \leq 2^{2s}$.

In other words, in every graph with n vertices there is either a clique or an independent set of size $\approx \frac{1}{2} \log_2 n$.

A lower bound?

To prove a lower bound, we need to "find" a graph without large cliques and independent sets. From above we know that the s can not be smaller than $\frac{1}{2} \log_2 n$. We do not know if this is always possible, but we can prove some non trivial bound. The best bound we know is with $s \approx 2 \log_2(n)$. This bound is proved via the probabilistic method, which does not provide an "explicit"[3] graph, but just shows that such a graph (in fact, most graphs have this property).

Explicit constructions

Here we provide some non trivial bound, with an "explicit" graph. If G is a union of n disjoint K_n's then it has n^2 vertices and every $n + 1$ vertices is not a clique or an independent set. That is, $R(s, s) \geq (s - 1)^2$.

Summary

So far we have shown

$$s^2 \lesssim R(s, s) \leq 2^{2s}$$

---

[3]There are several ways to define explicit. One is "constructible in polynomial time." Explicit constructions of pseudo-random objects is of great importance.

and said that the lower bound can be increased to roughly $2^{s/2}$. The true asymptotic

$$\lim_{s\to\infty} (R(s,s))^{1/s}$$

is not known (exercise: prove that the limit exists).

Ramsey-like examples

Ramsey theory is now very broad and has connections to basically all areas of mathematics, both as an influence and as a tool. It appear in number theory, geometry, computer science, logic, and more.

Geometry I   Here is an example from geometry of a Ramsey-like question (Erdos-Szekeres conjecture, or Happy End Problem [Esther Klein suggested problem, and later became Mrs. Szekeres]). For $n \geq 3$, let $ES(t)$ be the smallest number n so that in every set of n points in the plane in general position (no three points are on a line) there are t points that form a convex polygon. Erdos and Szekeres proved:

$$ES(t) \geq 2^{t-2} + 1.$$

It was recently proved after many years [Suk]:

$$ES(t) = 2^{t(1+o(1))};$$

here $o(1)$ is a term tends to zero as t tends to infinity.

Example [Klein]: $ES(4) = 5$. (i) Find an example showing $ES(4) > 4$. (ii) Consider five points. If the convex hull of the points has more than 4 vertices, we are done. Otherwise, it is a triangle ABC. The line through the other two points CE partition the triangles to two parts, one of which has four vertices.

Geometry II   Let $M(n)$ be the smallest integer so that for every set of at least M points in the plane there is an axis parallel rectangle containing exactly n of the points.

I was asked this question some time ago, and found an example that shows $M(85) = \infty$. That is, there is a configurations with 900 points so that no rectangle contains exactly 85 points.

5.7   Matchings

Think of n workers and n tasks, where each worker can perform some of the tasks. Encode this information in a bipartite graph, where edge $\{w, t\}$ means that worker w

can perform task t. Is there a way to distribute the tasks in an optimal way?

A matching in a graph G is a collection of disjoint edges. A perfect matching is a matching that contains all vertices. In the question above, we ask if the graph has a perfect matching.

We shall give a criterion that is equivalent to the existence of a perfect matching. The neighborhood of a set $U \subset V$ of vertices is

$$\Gamma(U) = \{v \in V : \exists u \in U \ \{u, v\} \in E\}.$$

If a bipartite $G = (V_0, V_1, E)$ has a perfect matching then what can we say on $|\Gamma(U)|$? It is at least $|U|$. We will show that this condition is sufficient as well.

We say that $M = \{e_1, \ldots, e_m\}$ is a perfect matching of V_0 if it is a matching and for every $u \in V_0$ there is $i \in [m]$ so that $u \in e_i$.

Theorem 60 (Hall). *G has a perfect matching of V_0 iff for every $U \subseteq V_0$ we have* $|\Gamma(U)| \geq |U|$.

Proof. This $\Rightarrow$ direction is simple. We prove the other direction by induction.

The base case is simple.

The induction step - there are two cases:

1. If for every $U \subset V_0$ so that $U \neq \emptyset$ and $U \neq V$ we have $|\Gamma(U)| > |U|$. Then let $v_0 \in V_0$ and $v_1 \in V_1$ so that $\{v_0, v_1\} \in E$. Let G' be the graph obtained by deleting v_0, v_1 from the graph. The graph G' is small, and satisfies the induction hypothesis, so it has a perfect matching for V_0'. We get a matching for V_0.

2. So, assume that there is a non-trivial U so that $|\Gamma(U)| = |U|$. Let G' be the induced graph on $U, \Gamma(U)$. By induction, there is a perfect matching for V_0' in G'. We need to extend this matching to all of V_0. Let G'' be the induced graph on $V_0 \setminus U, V_1 \setminus \Gamma(U)$. We claim that G'' also satisfies the induction hypothesis, so there is a perfect matching for V_0'', as needed. Indeed, if this is not the case, then there is U'' so that $|\Gamma_{G''}(U'')| < |U''|$ which means that

$$|\Gamma_G(U \cup U'')| = |\Gamma_G(U)| + |\Gamma_{G''}(U'')| < |U| + |U''|$$

a contradiction.

$\square$

There are many applications of Hall's theorem. Here is a simple one. A graph G is called d-regular, if all of its degrees are exactly d. Try to build such graphs.

Claim 61. *If G is bipartite and d-regular then the edges of G can be partitioned to d perfect matchings.*

Proof. The proof is by induction on d. For $d = 0$, the claim holds. Assume $d > 0$. We claim that G has a matching of V_0, which must be a perfect matching, since

$$d|V_0| = \text{number of edges} = d|V_1|.$$

We use Hall's theorem: if $U \subseteq V_0$ then

$$d|U| = \text{number of edge touching } U \leq d|\Gamma(U)|.$$

$\square$

Duality

Let G be a bipartite graph. There is a duality between matchings and vertex covers.

Definition 62. *A vertex cover is a set of vertices $C \subset V$ so that for every $e \in E$ there is $v \in C$ so that $v \in e$.*

Think of the graph as a computer net work. A vertex cover is a collection of computers that control all wires.

Theorem 63 (Konig). *In a bipartite graph, the maximum size of a matching is the minimum size of a vertex cover.*

The size of a vertex cover is at least the size of a matching, because a cover must contain a vertex from each edge in the matching. It remains to prove the other direction.

Proof. Let C be a vertex cover of minimum size. Let $C_0 = C \cap V_0$ and $C_1 = C \cap V_1$.

Let G' be the graph induced on C_0 and $V_1 \setminus C_1$. We claim that G' has a matching for C_0, using Hall theorem. Indeed, if $U \subset C_0$ so that $|\Gamma_{G'}(U)| < |U|$ then we can replace U by $\Gamma_{G'}(U)$ and get a smaller vertex cover than C.

Similarly, there is a matching of C_1, and we get a matching of size $|C|$. $\square$

This duality is part of a more general phenomenon; duality of linear programing. This has a geometric meaning, and is very useful in solving problems (several very sophisticated algorithms emerged from duality).

A specific example is the maximum flow on graphs, which is using duality equivalent to the minimum cut. Define what is a flow from s to t in a graph. Define the maximum flow: think of s as a source of water, on t as a sink, and of edges as pipes of capacity 1. What is the maximum amount of water that can flow from s to t? Define the minimum

$s - t$ cut. The theorem says that the min-cut is the max-flow. That the max-flow is at most the min-cut is easy.

It is also related to game theory, and the von Nuemann minimax theorem, which states a somewhat surprising result: Assume two-players play a deterministic game in which one of them wins. The players can toss coins to help them make a decision. Assume that for every strategy of player 2, there is a strategy of player 1 (he knows the strategy of player 1) in which he wins with probability at least p. Then, there is a strategy of player 1 (he does not know the strategy of player 2) that for every strategy of player 2, player 1 wins with probability p.

Chapter 6

Stable marriages

The last topic we discuss is related to stability and economics. A fundamental and important concept in the theory of economics is 'equilibrium'. Roughly speaking, an economy is stable if all of its parties do not have an incentive to change things.

We consider a model of marriages in society. There are n men and n women (for simplicity). Each person has an ordering of the n persons from the other group. The goal is to partition the $2n$ people to n pairs of man-woman.

Definition 64. *What makes such a partition stable? If every m and w that are not married are "stable", that is, either m prefers his wife over w or w prefers her husband over m. A marriage in which all pairs are stable is called stable.*

A natural question is

"does a stable marriage always exist?"

The american mathematicians David Gale and Lloyd Shapley answered this question in 1962.

Theorem 65 (Gale-Shapley)**.** *There is always a stable marriage.*

The Nobel Prize in Economics was awarded to Shapley in 2012 "for the theory of stable allocations and the practice of market design".

They actually explained how to (quite naturally) construct a stable marriage:
While there is a man m that is not engaged:

1. *m goes to the woman w that he likes most, among the women he did not propose to yet, and proposes to w.*

2. *If w is not engaged, she accepts.*

3. *If w is engaged to m', then*

   (a) *If w prefers m over m', she accepts.*

   (b) *Otherwise, she refuses.*

Correctness

The following properties hold (we shall not formally prove).

Claim 66. *From the moment a woman w is engaged she remains so.*

Claim 67. *Women only better their partners.*

Claim 68. *This procedure ends.*

Proof. If a man m is rejected by the last women on his list, this means that all women are engaged, which is a contradiction (number of men equals the number of women). The procedure ends in at most n^2 steps. $\qquad\square$

Claim 69. *When the procedure ends, we have a marriage.*

Claim 70. *This marriage is stable.*

Proof. If the marriage is not stable, there is (m, w) that is not stable. Namely, m prefers w over his wife w', and w prefers m over her husband m'.

   By construction, m offers to w before he offered to w'. When he offered to w, if she accepted then she would not accept m' offer later on. If she refused, she was married to a better person than m, and hence of m'. She would never end up marrying m'. $\qquad\square$

   You may think of other notions of stability. For example, you may ask for each person to marry the person he/she prefers most. This is not achievable.

   Interesting notions of stability combine "social interest" on one side and "realism" on the other.

Exercise 71. *Find an example with more than one stable marriage.*

Optimality

There could be more than one stable marriage in general. But this algorithm has an amazing property. It always produces the same marriage (regardless of the order of men), and moreover it is men-optimal! (In this version, the men did "all the work".)

Claim 72. *In every execution of this algorithm, the stable marriage that is produce is unique. In fact, each man is married to the best partner among all stable marriages.*

Proof. Let E be an execution of the algorithm. Let H be a stable marriage and H_E the output of E. For every (m, w) in H so that m prefers his wife in H over his wife in H_E, record the time that w refused or broke-up with m during E.

Assume that the set of such pairs is non-empty (towards a contradiction).

Let (m_0, w_0) be such a pair with minimum time.

Let w' be the wife of m_0 in H_E.

Since m_0 asked w_0 before w' during E, but ended up marrying w', we know that w_0 is not married to m_0 due to some specific m_1. And that w_0 prefers m_1 over m_0. (m_1 is not necessarily married to m_0; they were just engaged.)

Let w_1 be the wife of m_1 in H.

There are two cases:

1. If m_1 prefers w_1 over w_0: This means that prior to the engagement of m_1 and w_0, man m_1 asked w_1. This happened before m_0 was rejected by w_0, by the choice of m_1. Since m_1, w_1 are not married in H_E, this means that w_1 rejected m_1 before w_0 rejected m_0. A contradiction to the minimality.

2. If m_1 prefers w_0 over w_1: This is a contradiction to the stability of H, since w_0 also prefers m_1 over m_0.

$\square$